EDUC9970 Doctoral Research Project Plan

‘Doing maths’: The relationship between abductive and inductive actions and the development of quantitative reasoning in young students.

18 May, 2015

8,600 words

Michael Webb
18803659

Background and context – Algebra as a 'gatekeeper' to mathematics

Algebraic thinking is one of the most substantial areas of research in mathematics education (Watson, Jones and Pratt, 2013; Rakes, Valentine, McGatha and Ronau, 2010). The depth and breadth of research into algebra teaching and learning reflects both the high level of difficulty experienced by many students within the domain of algebraic thinking (Keiran, 2007), and its consequential "*gatekeeper*" (Cai and Knuth, 2011, p.6) role in restricting student participation in higher levels of school mathematics.

Definitions of algebra vary greatly. This diversity reflects the wide range of technological and cultural contexts of the subject, resulting in a broad body of research (Lins and Kaput, 2004; Keiran, 2007; Wilkie, 2014). Notwithstanding this epistemological challenge, a clear central element within most definitions of algebraic thinking can be seen in the construct of mathematical generalisation. Here, generalisation is defined to be "*the discovery, by reflection upon a number of cases, of a general description applicable to all of them"* (Peirce, 1960, p.256).

Within the Australian Curriculum, mathematical generalisation is situated within the proficiency of Reasoning. The interpretation and application of mathematical proficiencies is integral to a full understanding of the Australian Curriculum. Here, the four proficiencies - *Understanding*, *Fluency*, *Problem Solving* and *Reasoning* describe students' actions when engaging with the content areas of *Number and Algebra*, *Measurement and Geometry*, and *Statistics and Probability* (ACARA, 2015; ACARA, 2015b).

The proficiency of *Reasoning* is particularly important to the *Number and Algebra* content area of the Australian Curriculum. Here, the centrality of mathematical generalisation to the development of algebraic thinking is affirmed by the Australian Curriculum, Assessment and Reporting Authority (ACARA) who highlight the emphasis in the Pre-primary to Year 6 Number and Algebra Scope and Sequence of "*patterns that lead to generalisations*" (ACARA, 2015c). This approach to the development of algebraic thinking through pattern generalisation is regarded as a key curriculum focus for Australian upper primary students, and broadly mirrors sequences in the United Kingdom and parts of North America (Wilkie, 2014; Watson, 2009). In the Australian context, the Year 6 curriculum statements incorporate a significant change in the context of generalisation from that of number patterns, to the continuation, creation and description of rules relating to geometric (sometimes referred to as figural) patterns (ACARA, 2015).

Given this background, the objective of this paper is to locate research questions relating to the development of younger students' understanding of mathematical generalisation. This is achieved by initially considering literature that contextualises the development of students' understanding of mathematical generalisation as both a key element of, and as a pathway to, algebraic thinking. Subsequent to this, recent research on teacher pedagogical content knowledge is used to critically interpret the instructional practices that aim to develop an understanding of mathematical generalisation in Australian upper primary students. This interpretation is then contrasted with the key perspectives of Early Algebra literature, and in particular, the importance of mathematical structure and symbolism in the development of increasingly sophisticated understanding of mathematical generalisation in younger students.

Within the domain of Early Algebra, a key area of emergent research is that of instructional context. Specifically, research by Ellis (2007, 2007b, 2011) and Thompson (2011) identifies a number of issues relating to the efficacy of authentic quantitative contexts in the development of functional thinking in younger students. One such issue is the nature of the relationship between students' generalisation actions and the development of quantitative reasoning. This relationship provides a context for future inquiry that seeks to establish the efficacy of an instructional approach that is based on a technologically facilitated mathematical modelling cycle.

Literature describing the centrality of mathematical generalisation to the development of algebraic thinking

Mathematical generalisation is central to algebraic thinking. Rivera (2013) provides an extensive treatment of the multiple characterisations of mathematical generalisation, beginning by citing Peirce's (1960) conception of generalisation as the discovery of the general through the reflection on the specific. This idea of 'transformation through reflection' is formalised by Polya's (1957, p. 108) description of generalisation as being the "*passing from the consideration of one object to the consideration of a set containing that object; or passing from the consideration of a restricted set to that of a more comprehensive set containing the restricted one.*"

Definitions of generalisation can be extended to include the typology of associated thinking. Davydov (1990) and Kaput (1995) augment Peirce and Polya's ideas of generalisation as transformation through reflection by highlighting the significance of the action of classification, based on perceived, shared traits. Here, Davydov (1990) characterises the action of reflection by defining it to be an iterative cognitive cycle of abduction and induction, where the objective is the identification of invariant traits - or mathematical structures - across set members. In this context, abduction is defined in the Peircean sense to be the generation of a plausible inference based on a case, or series of cases, where this inference is subject to a subsequent inductive action of validation again new cases (Rivera and Becker, 2007). Kaput's (1995) treatment of generalisation shares Davydov's explicit focus on abducing mathematical structures, but is further extended to include the objective of developing the highest possible mathematical structural representation of the traits of set members, and in seeking to apply these structures as mathematical objects in order to generate new set members.

Abductive and inductive actions can also be seen as recursive cycles leading to generalisation. In this sense, Mason (2002) uses a falsificationist methodology to contextualise Kaput's (1995) ideas relating to the development of higher mathematical structures. Here, the function of generalisation becomes the application of the generalised mathematical object in seeking to generate variant set members, and thus falsify a hypothesised mathematical generalisation. This process, according to Rivera (2013, p.94), suggests that Mason's conception of generalisation necessitates a recursive cognitive cycle of "*an inductive action with an abductive intent*." That is, the failure of a mathematical structure to accommodate a marginal generated example signals the need for further abduction.

Finally, generalisation necessarily includes intersubjectivity. Here, Davydov (1990, p.175), emphasises a Vygotskian socio-cultural perspective by suggesting that the primary function of generalisation is to convey a mathematically useful structure through a system of "*sign-meaning-communication*". In characterising generalisation as a socio-cultural action, attention is placed on the importance of the forms of semantic and syntactic representations (including words, diagrams, argumentation, illustrations and symbols) that are reflective of socio-cultural norms and technological artefacts - both within a mathematics classroom and across broader society.

Given the various perspectives within the literature, this analysis adopts a definition that regards generalisation as a sequence of culturally situated cognitive actions. Specifically, generalisation is considered to be a socio-culturally and technologically mediated recursive cognitive process of abduction and induction, where the objective is the representation of the highest mathematical structure.

Literature describing the development of mathematical generalisation within the K-8 Australian curriculum

The Australian Curriculum's *Number and Algebra* Scope and Sequence is characterised by a significant emphasis on pattern generalisation as a pathway to understanding algebraic functions and expressions (Mulligan, Mitchelmore, English, and Creventsten, 2012). The key stages of this sequence can be better understood through the use of Smith's (2008) three-level descriptive framework as an analytical structure. Specifically, Smith's (2008) framework of pattern generalisation can be summarised as:

- Recursive thinking identifies variation or change within a sequence of quantities, numbers or values - e.g., '4, 8, 12, ___ '
- Co-variational thinking which identifies a coordinated change between two quantities (i.e., measured attributes) - e.g., 'my team's score increases by six points when we kick a goal'; and
- Correspondence thinking identifies and expresses an underlying dependent structure between two variables - e.g., 'my team's score can be described as 's = 6g+p'.

Applying Smith's framework of generalisation to the *Number and Algebra* Scope and Sequence highlights a progression from recursive to co-variational thinking in the primary years. That is, within Years 1-3, pattern generalisation outcomes are contextualised by an arithmetic focus on addition and subtraction within the context of numerical patterns - i.e., recursive pattern structures. With the arithmetical focus switching to multiplicative thinking in Years 4-6, number pattern generalisations allow for the construction of co-variational structures. At the Year 6 level, this multiplicative focus combines with the explicit introduction of geometric pattern contexts and a concomitant requirement to describe pattern generation rules. Here, the distinguishing characteristic of geometrical patterns – as "*having not one but many possible configurations or sub-configurations*" (Duval, 1999, p.17) - requires that students attend to higher levels of mathematical structure that can be expressed in multiple, mathematically equivalent forms, and therefore serves to introduce students to correspondence thinking.

A number of studies have highlighted the difficulties experienced by upper primary students in developing co-variational and correspondence thinking. Wilkie (2014) applies Smith's generalisation framework in reviewing research relating to pattern generalisation in an Australian primary setting. Specifically, studies highlighted by Wilkie (2014) include those of Stacey (1989), McGregor and Stacey (1995), English and Warren (1998) and Warren (2005). In general, these studies found that students experienced significant difficulty in progressing beyond recursive thinking. The sources of these difficulties were seen to relate to an inability to move beyond informal pattern description to the mathematisation of pattern rules, and corresponded with an over reliance on recursive strategies for pattern extension.

Warren and Cooper's (2008) important Australian study highlights the efficacy of explicitly introducing primary students to multiple forms of algebraic representation. In their three-year study of Year 3-5 students, Warren and Cooper (2008) found a predominance of recursive generalisation instructional content in the participant classrooms. This significantly limited the development of non-recursive generalisation thinking in students. In their teaching intervention, Cooper and Warren (2008) focused on developing student understanding and fluency with multiple representational forms. This approach provided students with the opportunity to identify commonalities across representations, thereby deepening both their conceptual understanding of mathematical structures, and simultaneously increased their repertoire of representational symbols and language (Watson, 2009).

Findings relating to the instructional efficacy of multiple representation and language are contextualised by research that highlights limitations in the pedagogical content knowledge of Australian teachers (Wilkie and Clarke, 2015). In a recent study of Australian teacher awareness of generalisation and reasoning in mathematics, Clarke, Clarke and Sullivan (2013) found that the verb ‘generalising’ was regularly used by just 35 per cent of surveyed Australian primary teachers (n=104). In general, Clarke, Clarke and Sullivan (2013) found that teachers required professional development and support in identifying learning tasks that motivate reasoning, and also need to be increasingly immersed in the language and concepts of mathematical reasoning.

A second Australian study highlights limitations in upper primary teachers’ pedagogical content knowledge relating to generalisation. Here, Wilkie (2014) reports that the majority of participant teachers were unable to generalise a geometric pattern to the correspondence level described in the Year 6 curriculum. Further, Wilkie (2014) suggests that teachers needed significant professional development to deepen their understanding of and instructional repertoire relating to non-recursive pattern generalisation.

Evidence of limitations in teacher pedagogical content knowledge relating to mathematical generalisation is clearly significant. However, difficulties in students’ progression from recursive generalisation to co-variational and correspondence thinking cannot be solely attributed to issues of teacher pedagogical content knowledge. International studies highlight limitations in the evidenced efficacy of pattern generalisation as an instructional pathway to algebraic thinking. Specifically, Moss and London McNab’s (2011, p.278) review of international literature relating to pattern generalisation understanding in older children found that the instructional potential of pattern generalisation pathways to algebraic thinking “*has not been substantially realised*.”

Instructional factors clearly mediate the efficacy of pattern generalisation as a pathway to algebraic thinking. These factors are likely to include the incorporation of appropriate mathematical structures in pattern design, teachers’ awareness of the developmental nature of students’ generalisations, enhancing students' representational repertoire (including language, symbols, tables and graphs), and the context of tasks (including diagrams, text, numbers and quantities) (Rivera, 2015). In this sense, Rivera (2015, p.187) recognises that instructional factors represent a form of "*cultural mediation"* that must be considered in order for pattern generalisation to be accessible for all learners.

Rivera's (2015) observations point to likely the significance of explicating useful mathematical structure in pattern generalisation instruction. Here, Ketterlin-Geller and Chard (2011) note growing evidence that focusing on conceptual development through explicit algebraic instruction will result in improved student outcomes. It is this objective that characterises the body of research referred to as Early Algebra - defined here as algebraic reasoning and instruction among learners between 6 and 12 years (Carraher and Schliemann, 2007, p.670).

Literature relating to Early Algebra and quantitative reasoning

According to Kaput and Blanton (2001, p.344), the objective of Early Algebra can be broadly regarded as efforts towards "*algebrafying the elementary mathematics experience.*" Within this broad objective, Kaput (2007, 2008) identifies two key perspectives of reasoning within Early Algebra research. These are a focus on the representation of generalisation through increasingly sophisticated and orthodox symbolism, and the significance of syntactically guided actions on symbols. Given this focus on symbolic representation and higher levels of generalisation, functional thinking can be clearly seen as central to Early Algebra perspectives. Here, functional thinking is defined to be the generalisation of a relationship between two variables or quantities (Keiran, 1996).

A number of studies have found that young students are able to demonstrate a developmental understanding of functional thinking. Specifically, studies by

Carraher, Schliemann, and Schwartz (2008), Mason (2008), and Cooper and Warren (2011) found that students in middle and upper primary grades can – with intervention - reason about functions and use words, diagrams and symbols to express recursive, co-variational and correspondence generalisations. In general, these studies highlight the important role of equipping younger students with a flexible representational repertoire (e.g., tables, function machines, language and diagrams) to interpret and express numerical relationships (Blanton and Kaput, 2011).

Cooper and Warren's (2011) Early Algebra Thinking Project study of Year 3 – 5 students is an important contribution to understanding the capacity of Australian primary students to develop functional thinking. A key finding of Cooper and Warren (2011) is that students' experienced significant difficulty in representing 'real-world' relationships as functions, and were generally unable to represent functional relationships algebraically. Cooper and Warren (2011, p. 201) attributed these difficulties to students' limited understanding regarding order of operations and a general tendency in classrooms to present 'real-world' problems as de-contextualised numerical statements or expressions.

Cooper and Warren's (2011) findings regarding students' difficulties in functional thinking are important in shaping the direction of future Early Algebra research. In particular, Cooper and Warren (2011) adopted an instructional approach that replaced a traditional presentation of functional thinking tasks where students are presented with tables of data and asked to abduce a generalised rule, with one based around function machines. Function machines incorporate a quasi-variable arithmetic generalisation that allows for the transformation of a numerical input to an output. In this context, students were asked to abduce the nature of the direct and inverse arithmetic generalisation. Clearly, task context and representation are significant factors in the development of young students' functional thinking.

Research highlighting the importance of multiple functional representations can be contrasted to work that emphasises learning context. Research by Confrey and Smith (1995), Smith and Thompson (2007), Ellis (2007b), Ellis (2011) and Thompson (2011) investigates the significance of functional thinking that is contextualised by quantitative referents. Specifically, these researchers highlight the efficacy of developing functional thinking that is contextualised by quantitative, rather than numerical inputs – i.e., quantitative reasoning. Within this context, quantities can be regarded as the measurements of the perceived attributes of an object or phenomena (Smith and Thompson, 2007).

Quantitative contexts can help students to build deeper conceptual understandings of functions. Ellis (2011), citing Confrey and Smith (1995), suggests that where students are required to engage in quantitative reasoning, this will typically be from the perspective of co-variational, rather than correspondence generalisation. Here, students are able to draw on their fundamental understandings of association, dependence and causation of phenomenon to support their development of mathematised relationship - e.g., the association between the duration of travel and displacement from origin (Ellis, 2007; Chazan, 2000). Whilst students would ultimately be expected to develop correspondence generalisation (e.g., displacement equals velocity by elapsed time), providing them with an understanding of co-variational generalisation can deepen students' understanding of algebra (Jacobson, 2014). That is, quantitative thinking and co-variational generalisation can "*serve as the true source and grounds for the development of algebraic methods*" (Smith and Thompson, 2007, p.96). Here, the situating of quantitative reasoning in authentic quantitative problem solving contexts can be seen to provide experientially real situations for students, rather than simply representing quantities as "*cover stories for proceduralized and frequently irrelevant tasks*" (English, 2003, p.5).

In addition to the significance of authentic quantitative referents to the deepening of conceptual understanding of functions, the use of quantities has the potential to activate the rich, concrete context of primary classrooms. At the primary level, the Australian Curriculum includes a significant emphasis on indirect and direct measurement of attributes – including length, area, volume, capacity, temperature, time, rotation and mass. These contexts provide an opportunity to introduce functional thinking in a "*rich, inquiry-based atmosphere*" (Blanton and Kaput, 2005, p.14) that should characterise a primary classroom. Finally, Ellis (2011, p.234) suggests that middle school students would benefit from the opportunity to explore the nature of linear relationships by "*directly manipulating quantities,"* - e.g., reasoning how changing the diameter of a wheel relates to the quantity of radius.

Despite the theoretically compelling arguments supporting the use of quantities as a context for the development of functional thinking, evidence from classroom-based studies provides only mixed support for this instructional approach. Specifically, studies by Ellis, (2007b), Lobarto and Siebert, (2002); and van Reeuwijk and Wijers, (1997) highlight a number of sources of difficulty, experienced by students, in learning to develop mathematically useful generalisations within the context of authentic quantitative referents.

One important factor identified by Ellis (2007) relates to the significance of student understanding of an authentic relationship. Specifically, where students had limited authentic experience in the nature of the quantitative relationship, they tended to revert to recursive pattern generalisation actions when considering quantitative data (Ellis, 2007). By way of contrast, students that demonstrated a fundamental understanding of a quantitative relationship, acquired through direct personal experience, were more likely to attempt to construct co-variational mathematical structures (Ellis, 2007). In the words of Thompson (2011, p.51), a mathematised model is a generalisation of *"one's understanding of a situation's inner mechanics – of "how it works*"." The importance of a shared, prior quantitative experience serves to highlight a potential role for assistive classroom technologies, and this is discussed later in this analysis.

A second factor that qualifies classroom based research support is the varied typology of individual's abductive approaches. Working with geometric pattern task items, Rivera and Becker (2007) found that individuals exhibited preferences towards either numerical or figural abduction strategies. These preferences significantly affected the process by which abduced forms are identified, mathematised and generalised in symbolic represented (Rivera and Becker, 2007, p.152). Preferences towards either a numerical or figural abduction approach were also found to be associated with the capacity of an individual to make judgements about the validity of the abduction, and to identify higher forms of abduction.

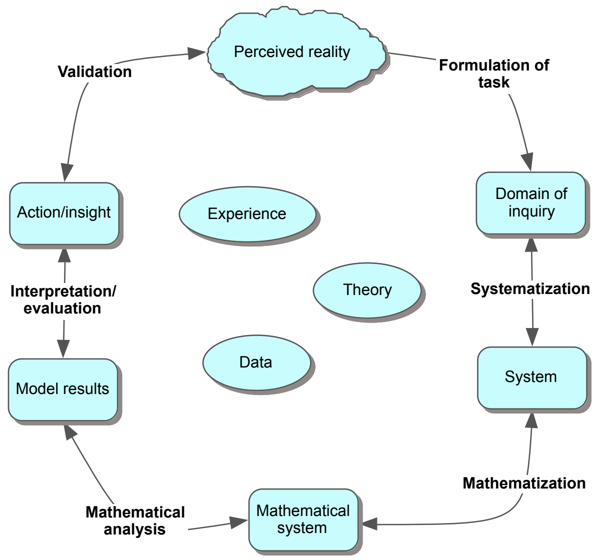
Finally, Thompson (2011) conjectures that the effectiveness of using directly manipulable quantities as a generalising context depends on a student's capacity to represent their abduced mathematical structure, and their ability to attend to the relevant attributes within a quantitative context. According to Ellis (2007), although quantitatively contextualised reasoning can support the increased sophistication of student generalisation, students who are unable to create mathematically useful structures may not gain any additional benefit from working with real-world quantities. Thus, the development of quantitative reasoning in students is likely to be facilitated through the explication of underlying mathematical structures and through attending to the development of students understanding of enabling quantitative operations. Here, Blanton and Kaput (2011, p.20) reason that the development of young students' functional thinking through instruction pre-supposes the existence of a supportive "*culture of practice*" within the classroom.

On balance, the literature suggests that the development of functional thinking in younger students is viable, and represents an alternative pathway to formal algebraic operations. However, instructional approaches and task contexts clearly play an important role in the development of functional thinking. The use of quantitatively contextualised tasks has the potential to deepen students' conceptual understanding of functions, however results from classroom-based trials have highlighted a number of instructional issues. These issues include the importance of an individual's level of conceptual understanding of the underlying quantitative relationship, and the significance of typological intra-individual differences in abductive actions. Finally, the importance of an instructional culture that explicates mathematical structure and operations is highlighted. It is here that Barbosa's (2006, p.297) description of mathematical modelling as a "milieu" of authentic mathematical inquiry is considered to be highly relevant.

Literature relating to mathematical modelling as an instructional approach.

Mathematical modelling can be defined as a bi-directional cycle of sub-processes, including the mathematising of a generalised structure (abduction), and concluding with a validation (inductive) action. (Frejd, 2013; Blomhoj and Kjeldsen, 2006; Cai et al., 2014). A schematic overview of a mathematical modelling cycle can be seen in Figure 1 below (Blomhoj and Kjeldsen, 2006, p.166).

Figure 1. Mathematical Modelling Representation



Modelling in mathematics education has attracted increased research attention over the past fifty years (Frejd, 2013). More recently, mathematical modelling has begun to feature in a number of secondary mathematics curricula (Blum, Galbraith and Niss, 2007). Contemporary literature contributions that consider the efficacy of modelling in mathematics education can be classified within three general categories – (1) the development of a mathematical disposition in students; (2) the development of students' modelling abilities; and (3) the development of students' mathematical concepts (Lehrer and Lesh, 2013).

There are a limited number of Australian studies of mathematical modelling instructional practices with young students. In English's (2003) study, a socio-cultural paradigm forms the basis of findings that primary students could be seen to develop increasingly generalised mathematical representations as they worked collaboratively to complete multiple mathematical modelling cycles. English (2003) further notes that the abstraction (i.e., the application of a mathematically generalised structure across novel, but similarly structured contexts (Australian Association of Mathematics Teachers, 2015)) of students' generalisations was critically facilitated through whole-of-class discussions. Mousoulides and English's (2006) study reports evidence that Year 6 students' demonstrated increased engagement in collaborative, evaluative discourse during the mathematising and validation stages of repeating mathematical modelling cycles.

From a utilitarian perspective, mathematical modelling can be regarded as a form of inquiry instruction. This conception of mathematical modelling can be distinguished from a normative treatment, where the instructional goal becomes the development of mathematical modelling competency in students. Perceiving mathematical modelling as a culture of mathematical inquiry has three significant implications for instructional practice relating to the development of functional thinking in younger students. According to Doerr and English (2003) and English (2006), these are considered to be (1) the recognition that the nature of quantities and operations must go beyond those typically considered in primary school mathematics activities; (2) that the use of authentic contexts can elicit the development of useful mathematical structures through mathematising sub-processes; and (3) that the inductive validation sub-process provides a purposeful context for the formalised representation of mathematical generalisations.

Beyond the role of authentic contexts, the eliciting of symbolic mathematical structures is further facilitated by the explicit treatment of mathematical notation, symbols and multiple modes of representation. Specifically, Blanton and Kaput (2011) identify the importance of introducing students to complex mathematical tools, new systems of notation and varied representational forms. In this sense, learning that results from an individual's cognitive internalisation of notation and symbolism can be seen to reflect the perspectives of Vygotsky's (1981) theory of socio-cultural cognition.

In summary, literature describes increased interest in mathematical modelling from the perspectives of both research and practice. The recursive abductive and inductive sub-processes within a modelling cycle may represent an instructional approach that assists younger students to elicit useful mathematical structures relating to authentic quantitative relationships. The abducing, induction and representation of these structures is likely to be facilitated by providing younger students with a broader repertoire of representational forms. However, the institution of a mathematical modelling cycle using authentic quantities together with the introduction of new forms of symbolic representation clearly represent significant challenges in a primary classroom setting. Given this, it is argued that emergent digital technology may provide an environment that supports the 'scaffolding' of students in their exploration of algebraic symbolism and in their direct manipulation of authentic quantities.

Literature relating to digital micro-worlds and robotics in education

Digital micro-worlds represent an instructional environment that supports student abstraction. In this sense, micro-worlds provide students with the opportunity to connect mathematical ideas with formalised mathematical representations (Noss and Hoyles, 1996). Here, classroom mathematical thinking becomes increasingly connected to "*official mathematical discourse*" (Sacristan et al., 2010, p.183). In a Vygotskian sense, digital micro-worlds can be analysed as an environment that provides students with the opportunity to gain mastery over cultural tools and symbols, thus supporting their cognitive development through a process of internalisation (Hogan and Trudge, 1999).

Work relating to the use of digital micro-worlds as a tool for the development of mathematical generalisation has a long lineage, extending back to the work of Papert (1980) and the development of *Logo*. *Logo* (and derived micro-worlds such as Massachusetts Institute of Technology's Scratch and Lego's Mindstorm environments) has been used as an instructional tool in mathematics for over thirty years. Conceived by Papert (1980) as a programming language that was accessible for young children, the *Logo* micro-world is structured within the mathematical context of Eucledian geometry, located in a Cartesian coordinate plane. Papert's (Papert and Harel, 1991, p.1) theoretical perspective in the development of *Logo* – constructionism - was an extension of Piaget's ideas of constructivism, and considers learning to be "*building knowledge structures, irrespective of the circumstances of learning*."

Given its long history of classroom application, there is an extensive range of literature that considers the use of Logo in the teaching of mathematics. Jones' (2005) review of Logo research, found evidence that Logo usage helped students to build connections between spatial and algebraic thinking. Further, whilst the general focus of Logo application has been in the content area of geometry, there was evidence that students were simultaneously found to develop algebraic thinking through the creation of coded procedures (Jones, 2005).

However, despite evidence of the efficacy of Logo based mathematics inquiry, research also identifies significant challenges to teaching methods and the sequencing of curriculum. Specifically, students were observed to frequently adopt an iterative, visual debugging of their coded constructions, without apparent reflection regarding the adequacy of their underlying mathematised structure (Watson, Jones, Pratt, 2013). Further to this, work by Pea (1987) found little evidence in support of the transfer of computational operations to other forms of mathematical and logic based learning activities. Importantly, the emergence of Vygotskian ideas of the role of social context and language as mediating tools in learning in the 1980's, posed significant challenges for the Piagetian pedagogies of play based learning that underpinned the development of many micro-words – including *Logo* (Sutherland, 1991; Healy and Kynigos, 2010).

Given this increased instructional emphasis on the role of context and language in mathematics classrooms, the criteria for the selection of an efficacious micro-world must now include elements of representation and communication. Specifically, the selection of a digital micro-world for the teaching of mathematics should consider (1) the generality of the tool (i.e., its applicability to different mathematical contexts); (2) its potential to support mathematisation (i.e., its capacity to express a mathematical representation); and (3) the communicative power of the tool (i.e., how relatable its symbol system to formalised mathematical symbolism) (Hershkowitz et al., 2002; Sacristan et al, 2010).

Applying these criteria to *Lego Mindstorms*, a derivative of Logo, is illustrative of changes to micro-worlds that reflect the role of symbols and communication (Ardito, Mosley and Scollins, 2014). In September 2013, *Lego Education* released a new version of *Mindstorms* micro-world, referred to as EV3. Changes to the original Logo programming system include the inclusion of standardised linear and non-linear functional structures, and the development of digital design journals. These digital design journals allow students to capture multi-media elements including video, images, sound, text, data and code. Pages within the design journal can be hyperlinked to other journal pages, thus theoretically supporting the presentation of a chronological or themed learning progression. It is considered that these digital design journals, when embedded within a form of mathematical modelling cycle, may provide an emergent means of observing the development of abductive and inductive mathematical reasoning in students together with recording evidenced examples of forms of student quantitative reasoning. The *Lego Mindstorms EV3* micro-world also provides a rich array of integrated robotic sensors and servos. Here, Black, Segal, Vitale and Fadjo (2012) report that programming robotics provided students with a form of ‘surrogate’ embodiment, enabling them to express and then observe quantitative relations, expressed through movement.

Research relating to the use of robotics in pre-college education is an emerging field. Benniti's (2012) meta-study of educational robotics reviewed ten studies that included quantitative assessment of student learning in a primary or secondary setting. In general, the context of the research involved forms of integration across the domains of mathematics and physical sciences (Benniti, 2012). Subsequent to Benniti's (2012) review, Ardito, Mosley and Scollins (2014) undertook a mixed methods intervention study that considered the effect of a sequenced robotics program on the development of Year 6 students' general mathematical performance and the nature of learning collaboration within a classroom. In general, this study found significant increases in the level of group affiliation of students in the mathematics classroom along with increases in conceptual understanding of several mathematical correspondence representations of circle area and circumference. There are no identified studies that consider robotics as a domain of abstraction that supports the development of abductive/inductive thinking and their relationship to the development of quantitative reasoning in younger students.

Mathematical generalisation is central to the development of mathematical reasoning and algebraic thinking. Research into the development of mathematical generalisation in younger students supports the consideration of alternative instructional approaches to the pattern generalisation sequence that characterises the Australian Curriculum. Here, Early Algebra research provides a context for the development and testing of theory related to algebraic teaching and learning for students between the ages of 6 and 12.

Within Early Algebra research, functional thinking and symbolism are central to the development of algebraic reasoning. The use of authentic quantitative contexts may provide students with an opportunity to draw on their existing understandings of association, causation and dependency, in abducing mathematical structures. However, mixed results from classroom based experiments involving the use of quantitative contexts highlights the importance of a better understanding of both the typology of abductive and inductive actions, and the nature of their interaction in the development of quantitative generalisations (Ellis, 2007).

Research relating to the role of abductive and inductive thinking leading to the development of quantitative generalisations by young students is limited. Here, the work of Rivera and Becker (2007) provides some guidance in terms of the significance of abductive approaches, however this study was conducted with 34 pre-service elementary teachers, and not upper primary students. In addition, Becker and Rivera's (2007) tasks were examples of geometric pattern, and not quantitative relationships. Ellis' (2007) study highlights the likely significance of a students' prior understanding of a quantitative relationship in influencing their capacity to abduce quantitative relationships. However this study was primarily focused on contrasting the development of generalisation across numerical and quantitative contexts, and was not specifically focused on the characteristics of students' abduction and induction actions within a singular context.

Research paradigm - Vygotskian socio-cutural knowledge

Limitations in existing research motivate new questions regarding the connection between types of reasoning and generalisation in young students. Here, an important objective of future research is in establishing new theory regarding the typology and interaction of upper primary students' culturally mediated abductive and inductive actions that are associated with the development of quantitative reasoning.

The literature considered in this analysis provides a basis for understanding mathematical generalisation to be a socio-culturally mediated cognitive process of abduction and induction, leading to the communication of the highest mathematical structure. In this regard, the construct of mathematical generalisation adopted in this study is Vygotskian. Here, learning is constituted as the mediation of symbols, tools, peers and teachers are constitutive of learning (Lerman, 1996, p. 147). In the words of Vygotsky (1978, p. 56), internalisation is "*the internal reconstruction of an external operation*" and is motivated by socially shared experiences. Therefore, the research paradigm considers mathematical knowledge to be internalised cultural knowledge.

Research questions:

Therefore, the objective of this intended study is to generate new theory about the relationship between upper primary students' abductive and inductive thinking and the development of quantitative reasoning. This objectives lead to the following research questions:

1. Can upper primary students learn to generalise quantitative relationships?
2. In what ways do upper primary students' generalisations of quantitative relationships change over the course of a quantitative thinking instructional sequence?
3. What are the characteristics of upper primary students abductive and inductive actions as they generalise a quantitative relationship?
4. In what ways do robotic micro-worlds facilitate the abductive and inductive generalising actions of upper primary students?

Research Design – Design Based Research

Limitations in the existing body of Early Algebra literature motivate two methodological considerations for future research. Firstly, the typology and interrelationship between abductive and inductive actions associated with the development of quantitative reasoning has received limited attention in the context of an upper primary classroom. Thus, there is a need for new 'domain-specific' (diSessa and Cobb, 2004) theory in this area. Secondly, the emergent nature of educational robotics and mathematical modelling instruction suggest that their intersection as instructional approach is not reflective of current practice. This necessitates a research methodology that is experimental. Given these two considerations, it is argued that Design Based Research represents an appropriate methodology that would enable the formulation of an empirical response to the identified research questions.

Design Based Research

Design Based Research (DBR) is a relatively new educational research methodology (Anderson and Shattuck, 2012) that is located within the general domain of experimental research methods (Collins, Joseph and Bielaczyk, 2004). Cobb, Stephan, McClain and Gravemeijer's (2001) description of DBR is as a repeating cycle of instructional design (informed by an existing domain-specific instructional theory), and classroom based analysis (that is guided by an interpretive framework). Formally, Artigue (2015, p.469) defines DBR to be "*a formative approach to research, in which a product or process (or 'tool') is envisaged, designed, developed and refined through cycles of enactment, observation, analysis and redesign, with systematic feedback from end users.*"

According to Bakker and van Erde (2015), the recursive nature of DBR is a distinguishing methodological characteristic. Within the cycles of DBR, learning objectives are initially formulated into the design of a curriculum, known as a Hypothetical Learning Trajectory (HLT), that can be adapted during the empirical stages of a study.

The development of a HLT is fundamental to the DBR method. According to Simon (1995), the HLT is constituted of three components – (1) a statement of learning goals, (2) a sequence of learning activities, and (3) a theoretical learning process - i.e., a prediction of how students' understanding will develop over the course of the learning sequence. The HLT then serves as the central construct throughout the three stages of DBR – i.e., the design, experiment and reflective analysis phases.

Phase 1 – Design of a Hypothetical Learning Trajectory

The first phase of DBR involves the development of an HLT. This learning sequence is contextualised by existing domain-specific learning theory. In the case of this study, this is considered to be Early Algebra and more specifically, quantitative reasoning. The development of the HLT then involves a form of thought experiment that anticipates how students' mathematical actions and discourse may develop as they participate in the hypothesised learning sequence (Freudenthal, 1991; Cobb, Stephan, McClain, and Gravemeijer, 2001).

Given the research questions identified in this analysis, there are three important observations from literature that would be expected to influence the structure of an HLT. Firstly, where students are provided with a shared, rich prior experience in a quantitative relationship, it can be argued that that Ellis (2007) and Thompson's (2011) source of ambiguity relating to a fundamental quantitative relationship – i.e., how a process actually works - may be reduced across a class of students. Secondly, adopting an instructional approach that promotes the explication of students' abductive and inductive actions would allow for a more granular analysis of the relationship between these forms of reasoning, and generalisation. Thirdly, embedding the research jointly within a quantitative context and a classroom of upper primary students would overcome the limitations of Rivera and Becker's (2007) study.

Given the above, an HLT would incorporate unmediated student quantitative experiences within a mathematical inquiry structure. Specifically, developing an instructional sequence based on a form of mathematical modelling cycle would allow for the observation of both the typology and interrelationship of students' abductive (mathematising) and inductive (validating) actions. This, when combined with the collection of students' quantitative reasoning artefacts, would allow for the development of theory relating to the relationship between forms of student cognitive action, and the development of their quantitative reasoning. Given this, emergent robotics environments that include multi-media design journals might support students' reflective actions and allow them to communicate their abduced mathematical structures at key stages of a mathematical modelling cycle. Importantly, the multi-media capacity of the Mindstorms EV3 digital design journal allows for the recording of reflective discourse through the presentation of robotics coding, video (voice and gesture), image and text. Further, the abstractive nature of micro-worlds allows for students to be immersed in a learning environment that incorporates orthodox mathematical symbols and structures.

The duration and fidelity of the HLT would be informed by ongoing data analysis. The HLT would be expected to incorporate a series of mathematical modelling tasks, designed to develop students' co-variational and correspondence generalisation of quantitative relationships. It is expected that the approximate duration of the HLT sequence would be 10 weeks at a fidelity of 2 hours per week. It is estimated that between 3 and 4 DBR instructional cycles would be conducted. However, the number of teaching experiment cycles would ultimately be determined by methodological considerations (see the following section for a formal treatment of this issue), and in particular, the achievement of a point of theoretical saturation – i.e., where new data ceases to yield information regarding the conceptualistion of coded categories.

Phase 2 – Repeating teaching experiments and HLT adaptations

The nature of the HLT formulation is such that it represents only a tentative representation of the actual sequence of learning activities. In this sense, the second phase of the DBR cycle involves the review and necessary adaptation of the HLT, where this is informed by the continuous analysis of actual classroom events (Cobb, Stephan, McClain and Gravemeijer, 2001). HLT adaptations would be performed in consultation with the participating classroom teachers during each DBR cycle. These discussions would be initiated by reflective questions, video recorded and subjected to retrospective qualitative analysis. All HLT changes would be documented to allow for retrospective qualitative analysis.

Phase 3 – Retrospective analysis and theory generation

The DBR cycle concludes with an inductive process whereby students' learning outcomes are assessed against the objectives identified in the HLT. In the case of this study, the HLT learning objectives will address the question of: 'Can upper primary students learn to generalise quantitative relationships?" Here, students' learning artefacts will be compared to the HLT over the course of the DBR, with differences analysed through the development of a quantitative data analysis matrix (cf. Dierdorp, Bakker, Eijkelhof, and Van Maanen, (2011)).

Retrospective qualitative analysis is formative within a DBR. Here, student digital journals, together with documented HLT adaptations collected during the instructional sequence, will be retrospectively analysed – through an interpretative analytical framework – in order to generate new theory regarding student learning (Cobb, Stephan, McClain and Gravemeijer, 2001; Bakker and van Erde, 2015). It is this contrastive, retrospective analysis that provides for the generation of domain-specific instructional theory. In the context of this study, retrospective analysis of qualitative data obtained during each DBR cycle will be conducted through the application of the constant comparative method (Glaser and Strauss, 2009).

<u>Validity and reliability</u>

Context is central to the naturalistic setting of DBR. The use of authentic classroom based analysis in DBR results in the researcher having little control over the cultural context, and therefore this limits the replicability of a study. Given this, Barab and Squire (2004, p.10) suggest that the goal of DBR must be to "*lay open and problematize the completed design"* and its implementation so that insight is provided into local contexts. This approach enhances the external validity of a study by demonstrating how the findings can be informative in other contexts (McKenney and Reeves, 2013, p. 8). In addition to addressing the replicability of the study, the internal validity - i.e., "*the validity of a research design*" (Punch, 2009, p.360) – will be enhanced through the testing of initial theoretical conjectures against subsequent DBR teaching experiment cycles.

Reliability in qualitative research refers to the dependability of data and the objectivity of analysis (Punch, 2009, p. 359; Bakker and van Erde, 2015). Here, external reliability relates to the replicability of a study – that is, the independence of the study findings from the perspectives of the researcher (Bakker and van Erde, 2015). External reliability will be enhanced in this study through four strategies – (1) the maintenance of a strict delineation in the role of the researcher within the classroom; (2) the video recording and transcription of teacher responses to HLTs; (3) full publication of the final form of the HLT along with tasks and examples of student coding responses; and (4) the use of an internationally standard robotics software environment.

Finally, internal reliability – i.e., data and analytical reliability – will be enhanced through the adoption of the following protocols – (1) the archiving of codes versions at each DBR cycle; (2) video recording of student reflections and use of authentic student code as primary data; and (3) the use of independent sample coding and the estimation of inter-coder reliability through the calculation of Kronbach's alpha.

Participants

The DBR study will be conducted in authentic classroom settings. It is anticipated that between 3 and 5 classroom teachers will participate in the delivery of the HLT. Alongside the delivery of the instructional activities, these teachers will participate in a number of video interviews, leading to the adaptation of the HLT. Participant teachers will also complete the task specific summative assessment rubrics at the end of each of the HLT's quantitative modelling tasks.

Each class is expected to contain between 25 and 32 upper primary students, attending Western Australian metropolitan public schools. For this study, the term 'upper primary' is considered to be students currently undertaking studies at the Australian Curriculum Year 5 and 6 level. This translates to an expected chronological age of between 9 and 12 years of age. Importantly, Western Australian public primary schools are inclusive of students with special needs. It is therefore anticipated that one source of HLT adaptation will be to ensure the appropriate differentiation of the instructional sequence to meet the particular needs of a student. These needs would be established in discussions with the participant classroom teacher during the HLT development stages. However for ethical reasons, neither the identity nor specific learning needs of any individual student will be reported during the study.

The researcher will be a participant during each of the teaching experiment phases. The role of the researcher will primarily be that of an observer, however – the researcher will also provide technical information technology and robotics support to the participant teacher at their request. The researcher will not deliver instructional content, nor will they provide any form of coding assistance to students.

Finally, in order to enhance internal reliability, an experienced upper primary teacher will be recruited to fulfil the role of an independent code reviewer. This code reviewer will be trained in the operation of Atlas.ti coding software, and then proceed to code sampled episodes against established categories. The quantum of the code sample will be determined by the calculation of Kronbach's alpha, and will thus be jointly determined by the number of code categories, and the number of coded episodes. It is expected that this individual would be financially compensated for their involvement in the research project.

Data collection

The following quantitative and qualitative data will be collected for analysis during the course of the project.

Qualitative Data:

1. HLT Versions:
    a. Versions of the HLT will be archived throughout the multiple cycles of the DBR. These artefacts will take the form of learning goal statements, lesson plans, assessment rubrics, student handouts and Power-point slides. In addition, the HLT will include Lego Mindstorm EV3 Teacher Projects – including model building instructions, task descriptions and video

reflection stimulus questions. These items will be in the form of digital images, hyperlinked playbooks, video and text.

2. HLT adaptation memorandums and field observation notes:
    a. The HLT will be adapted during the empirical stage of the study. Each HLT change will be accompanied by the researcher's text memorandum that explains the basis of the adaptation.
    b. The researcher will be present in each participant classroom acting as an observer and robotics technical assistant. Observation notes relating to the HLT and any technical interventions will be retained for analysis.

3. Video interview with participant classroom teacher:
    a. The experiences of the participating classroom teacher in delivering the HLT are critical to its adaptation. Therefore, participant teachers will participate in a short video interview at the completion of each HLT quantitative modelling project.

4. Students' authentic robotics code:
    a. Students will develop their robotics code within the Lego Mindstorms EV3 project application. Each quantitative modelling task will require the creation of a Lego Mindstorms EV3 project file. This file will include versions of student code according to a protocol. This protocol will reflect the mathematising and validation sub-processes in the quantitative modelling instructional cycle, with students being required to save a version of their code each time they bring their robot to the central robotics map for testing. Versions of this code development can then be analysed in terms of their development towards generalised mathematical structures.

5. Students' digital multi-media design journals:
    a. The *Lego Mindstorms EV3* robotics software supports the collection of student generated multi-media artefacts. Here, learning artefacts would then be in the form of student digital journals that incorporated each version of authentic code, annotated with short, reflective video entries that related to each mathematising and validation sub-process. It is noted that Bakker (2004) found that the use of student 'mini-interviews' (between 20 – 40 seconds) to be particularly useful source of qualitative data in research relating to symbolism in a digital learning environment.

Quantitative Data

6. Task specific summative teacher assessed rubrics:
    a. At the end of each quantitative modelling task, the participant classroom teacher would complete a binary summative rubric for each pair of participating students. This rubric will include statements about student success at achieving each of the levels of an escalating sequence of robotics subtasks. The use of a binary scoring system will allow teachers to identify student success, but will not attempt to separate students code, mathematising or validation actions. This data will be used to address the overall learning outcome statements of the HLT, where these relate to the demonstration of co-variational reasoning through symbolic representation. Importantly, this code will also be used to triangulate the open coding of student video responses and robotics code.

Data analysis – constant comparative method

The analysis of the projects quantitative data will be conducted through open coding within the constant comparative method. In this sense, analysis involves the rendering of qualitative data into codes and categories that reflect layers of abstraction based on relations within the data (Teppo, 2015). There are three principal reasons for the selection of the constant comparative method in this project – (1) theory generation, (2) sampling, and (3) reliability.

Firstly, constant comparative analysis is a generative method. The inductive perspective of the constant comparative analysis can be seen in Glaser and Strauss' (1967, p.23) statement that – "*In discovering theory, one generates conceptual categories or their properties from evidence; then the evidence from which the category emerged is used to illustrate the concept*." Consistent with the generative perspective of constant comparative method, qualitative data will be subjected to an open coding process using Atlas.ti qualitative analysis software. A coding cycle will correspond with the completion of each HLT sub-cycle. This position corresponds with Glaser and Strauss's perspective that "*the development of theoretical constructs should occur simultaneously with data collection and analysis*" (Cobb and Whitenack, 1996, p.224).

Secondly, theoretical saturation provides critical guidance as to the number of sub-cycles necessary for a DBR study. The adaptive nature of the HLT accords well with theoretical sampling within the constant comparative method. Here, Strauss and Corbin's (1990, p.177) description of theoretical sampling as being "*sampling on the basis of concepts that have proven theoretical relevance to the evolving theory*" accords with the formative nature of DBR. Finally, the number of DBR redesign cycles is determined by the achievement of theoretical saturation.

Thirdly, the constant comparative method enhances the reliability of a qualitative study. In this sense, Cobb, Stephan, McClain and Gravemeijer (2010, p.517) note that this method of analysis ensures that the interpretation of any single episode is "*located within a network of mutually reinforcing inferences that span the entire data set*". The completion of each HLT sub-cycle will initiate a recoding of all the previously coded episodes. According to Cobb and Whitenack (1996, p.224), "*it is in the process of reconciling provision categories with subsequent data that newly formulated categories become stable and evolve into explanatory constructs*."

Ethical approach

Ethics approval will be sought from the University of Western Australia's Human Ethics office prior to the commencement of the research phase of this project. In obtaining ethics approval, particular consideration will need to be given to the issues relating to the participation of children. Ethical considerations relating to the participation of children in educational research are detailed in the National Statement on Ethical Conduct in Human Research (NSECHR) (Australian Research Council, 2015), and include the expected dimensions of research merit and integrity, justice, beneficence, respect, consent and welfare.

However, research that involves the participation of children requires specific strategies relating to informed consent. Given the age of the participant groups, it is considered appropriate that a multi-level cascading approach to consent be taken, initially involving school administration, then teachers, parents/guardians and the individual students. The NSECHR's (Australian Research Council, 2015) provisions for research with children requires that children and parents consent only where they have been provided with relevant information and have participated in developmentally appropriate discussions relating to the objectives and processes of the research project. Given this, the process of obtaining consent will necessarily involve information and discussion sessions for participant teachers, students and their parents/caregivers.

Finally, provisions would need to be made within the ethics protocols for students that did not consent to participate in whole of class research project. Ethical considerations would clearly preclude the removal of a non-participant child from the social setting of their classroom. Therefore, alternative educative activities would need to be negotiated with the participant classroom teacher that would ensure that non-consenting students were included in related activities, but were distinguished for the purposes of their exclusion from data collection.

References

Amit, M., & Neria, D. (2013). Assessing a Modelling Process of a Linear Pattern Task. In R. Lesh, R. Galbraith, P. Haines, & A. Hurford (Eds.), *Modeling Students' Mathematical Competencies ICTM 13*. Netherlands: Springer.

Anderson, T., & Shattuck, J. (2012). Design-Based Research A Decade of Progress in Education Research. *Educational Researcher*, *41*(1), 16-25.

Ardito, G., Mosley, P., & Scollins, L. (2014). WE, ROBOT: Using Robotics to Promote Collaborative and Mathematics Learning in a Middle School Classroom. *Middle Grades Research Journal*, *9(3)*, 73-88.

Artigue, M. (2015). Perspectives on Design Research: The Case of Didactical Engineering. In A. Bikner-Ahsbahs, C. Knipping, & N. Presmeg (Eds.), *Approaches to Qualitative Research in Mathematics Education*. Netherlands: Springer.

Australian Association of Mathematics Teachers (2015). *Abstraction and Generalisation*. Retrieved from http://www.aamt.edu.au/Topdrawer/Patterns/Good-teaching/Abstraction-and-generalisation

Australian Curriculum, Assessment and Reporting Authority. (2012). *Australian Curriculum: Mathematics – Framing Paper*. Retrieved from http://www.acara.edu.au/verve/_resources/National_Mathematics_Curriculum_-_Framing_Paper.pdf

Australian Curriculum, Assessment and Reporting Authority. (2015). *Australian Curriculum: Mathematics – Content Structure*. Retrieved from http://www.australiancurriculum.edu.au/mathematics/content-structure

Australian Curriculum, Assessment and Reporting Authority. (2015b). *Australian Curriculum: Mathematics – Curriculum.* Retrieved from http://www.australiancurriculum.edu.au/mathematics/curriculum/f-10?layout=1#level6

Australian Research Council. (2015*). National Statement on Ethical Conduct in Human Research 2007 (Updated May, 2105*). Retrieved from http://www.nhmrc.gov.au/_files_nhmrc/publications/attachments/e72_national_statement_may_2015_150514_a.pdf

Bakker, A., & van Erde, D. (2015). An Introduction to Design-Based Research with an Example From Statistics Education. In A. Bikner-Ahshbahs, C. Knipping, & N. Presmeg (Eds), *Approaches to Qualitative Research in Mathematics Education: Advances in Mathematics Education.* Netherlands: Springer.

Barab, S. & Squire, K. (2004). Design-Based Research: Putting a Stake in the Ground. *The Journal of Learning Sciences*, *13(1),* 1-14.

Barbosa, J. C. (2006). Mathematical modelling in classroom: a socio-critical and discursive perspective. *ZDM*, *38*(3), 293-301.

Bakker, A. (2004a). *Design research in statistics education: On symbolizing and computer tools* (Doctoral dissertation). Retrieved from http://dspace.library.uu.nl/bitstream/handle/1874/893/full.pdf

Bell, A. (1995). Purpose in school algebra. *Journal of Mathematical Behaviours*, *14*, 41-73.

Benitti, F. (2012). Exploring the educational perspective of robotics in schools: A systematic review. *Computers & Education, 58*, 978-988.

Black, J., Segal, A., Vitale, J., & Fadjo, C. (2012). Embodied cognition and learning environment design. In D. Jonassen & S. Lamb (Eds.), *Theoretical foundations of student-centered learning environments*. New York: Routledge.

Blanton, M., & Kaput, J. (2005). Characterizing a Classroom Practice That Promotes Algebraic Reasoning. *Journal for research in mathematics education*, *36*(5), 412-446.

Blomhoj, M., & Kjeldsen, T. (2006). Teaching mathematical modelling through project work. *ZDM*, *38*(2), 163-177.

Blum, W. (1994). Mathematical modeling in mathematics education and instruction. In T. Breiteig, I. Huntley & G. Kaiser-Messner (Eds.), *Teaching and learning mathematics in context.* London: Ellis Horwood Limited.

Blum, W., Galbraith, P., & Niss, M. (2007). Introduction. In W. Blum, P. Galbraith, H. Henn, & M. Niss (Eds). *Modelling and applications in mathematics education. The 14th ICMI study*, New York: Springer.

Boaler, J. (2002). Learning from teaching: Exploring the relationship between reform curriculum and equity. *Journal for Research in Mathematics Education, 33(4)*, 239-258.

Cai, J., Cirillo, M., Pelesko, J. A., Ferri, R. B., Borba, M., Stillman, G., English, L., Wake, G., Kaiser, G., & Kwon, O. (2014). Mathematical Modeling in School Education: Mathematical, Cognitive, Curricular, Instructional, and Teacher Education Perspectives. In P. *Liljedahl, C. Nicol, S. Oesterle, & D. Allan (Eds.), Proceedings of the 38th Conference of the International Group for the Psychology of Mathematics Education and the 36th Conference of the North American Chapter of the Psychology of Mathematics Education (Vol. 1).* Vancouver: PME.

Cai, J., & Knuth E. (2011). Introduction. In K. Cai and E. Knuth (Eds.), *Early Algebraization*. Netherlands: Springer.

Carraher, D., & Schliemann, A. (2007). Early Algebra and Algebraic Reasoning. In F. Lester (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning: Volume 2*. New York: NCTM.

Carraher, D., & Schliemann, A. (2014). Early Algebra Teaching and Learning. In S. Lerman (Ed.), *Encyclopedia of Mathematics Education*. Netherlands: Springer.

Carraher. D., Schliemann, A., & Schwartz, J. (2008). Early algebra does not equal algebra early. In J. Kaput, D. Carraher, & M. Blanton (Eds.), *Algebra in the Early Grades.* New Jersey: Erlbaum.

Chazan, D. (2000). *Beyond formulas in mathematics and teaching: Dynamics of the high school algebra classroom*. Teachers College Press.

Clarke, D. M., Clarke, D.J., & Sullivan, P. (2012). Reasoning in the Australian Curriculum: Understanding its meaning and using the relevant language. *Australian Primary Mathematics Classroom, 17(3)*.

Cobb, P., Stephan, M., McClain, K., & Gravemeijer, K. (2001). Participating in Classroom Mathematical Practices. In E. Yackel, K. Gravemeijer, & A. Sfard (Eds.), *A Journey in Mathematics Education Research.* New York: Springer.

Cobb, P., & Whitenack, J. (1996). A method for conducting longitudinal analyses of classroom videorecordings and transcripts. *Educational studies in mathematics*, *30(3)*, 213-228.

Collins, A., Joseph, D., & Bielaczyc, K. (2004). Design Research: Theoretical and Methodological Issues. *The Journal of Learning Sciences, 13(1)*, 15-42.

Confrey, J., & Smith, E. (1995). Splitting, covariation, and their role in the development of exponential functions. *Journal for Research in mathematics* education, 26(1), 66-86.

Cooper, T. & Warren, E. (2008). The effect of different representations on Years 3 to 5 students' ability to generalise. *ZDM, 40,* 23-27.

Cooper, T., & Warren, E. (2011). Year 2 to 6 Students' Ability to Generalise: Models, Representations and Theory for Teaching and Learning. In J. Cai & E. Kenneth (Eds.), *Early Algebrization: Advances in Mathematics Education*. New York: Springer.

Corbin, J., & Strauss, A. (1990). Grounded theory research: Procedures, canons, and evaluative criteria. *Qualitative sociology*, *13*(1), 3-21.

Davydov, V. (1990). Type of generalization in instruction: Logical and psychological problems in the structuring of school curricula. In J. Kilpatrick (Ed.), *Soviet studies in mathematics education: Volume 2.* Virginia: NCTM.

Day, L. (2014). Australian Curriculum linked lessons: Reasoning in Number and Algebra. *Australian Primary Mathematics Classroom, 19(3)*, 16-19.

Dierdorp, A., Bakker, A., Eijkelhof, H., & Van Maanen, J. (2011). Authentic practices as contexts for learning to draw inferences beyond correlated data. *Mathematical Thinking and Learning, 13*, 132–151.

diSessa, A. & Cobb, P. (2004). Ontological innovation and the role of theory in design experiments. *Educational Researcher, 32(1)*, 77–103.

Doerr, H. & English, L. (2003). A Modeling perspective on students' mathematical reasoning about data. *Journal of Research in Mathematics Education, 34(2)*, 110-136.

Dorfler, W. (1991). Forms and means of generalization. In A. Bishop & S. Mellin-Olsen (Eds.), *Mathematical knowledge: Its growth through teaching.* Netherlands: Kluwer.

Dorfler, W. (2008). En route from patterns to algebra: Comments and reflections. *ZDM, 40 (1)*, 143–160.

Duval, R. (1999). Representation, vision, and visualization: Cognitive functions in mathematical thinking. In F. Hitt & M. Santos (Eds.), *Proceedings of the 21st North American PME Conference.* Mexico: PMENA.

Ellis, A. (2007). The influence of reasoning with emergent quantities on students' generalizations. *Cognition and Instruction*, *25*(4), 439-478.

Ellis, A. (2007b). Connections between generalizing and justifying: Students' reasoning with linear relationships. *Journal for Research in Mathematics Education*, 38 (3), 194-229.

Ellis, A. (2011). Algebra in the middle school: Developing functional relationships through quantitative reasoning. In G. Kaiser & B. Sririman (Eds.), *Early Algebraization.* Berlin: Springer.

English, L. (2003). Modelling with young learners. In S. Lamon, W. Parker and S. Houston (Eds.), *Mathematical Modelling: A way of life.* Elsevier.

English, L. (2006). Mathematical modeling in the primary school: Children's construction of a consumer guide. *Educational Studies in Mathematics*, *63*(3), 303-323.

English, L., & Warren, E. (1998). Introducing the variable through pattern exploration. *The Mathematics Teacher, 91*, 166–170.

Frejd, P. (2013). Modes of modelling assessment—a literature review. *Educational Studies in Mathematics*, *84*(3), 413-438.

Freudenthal, H. (1991). *Revisiting mathematics education: China lectures.* Kluwer: Dordecht.

Glaser, B., & Strauss, A. (2009). *The discovery of grounded theory: Strategies for qualitative research*. Transaction Publishers.

Gravemeijer, K. (1994). *Developing realistic mathematics instruction.* Netherlands: Freudenthal Institute.

Harel, G. (2001). The development of mathematical induction as a proof scheme: A model for DNR-based instruction. In S. Campbell & R. Zazkis (Eds.), *Learning and teaching number theory: Research in cognition and instruction.* Connecticut: Greenwoood Press.

Healy, L., & Kynigos, C. (2010). Charting the microworld territory over time: design and construction in mathematics education. *ZDM*, *42*(1), 63-76.

Hershkowitz, R., Dreyfus, T., Ben-Zvi, D., Friedlander, A., Hadas, N., Resnick, T., & Schwarz, B. (2002). Mathematics curriculum development for computerized environments: A designer-researcher-teacher-learner activity. In L. English & Kirshner, D. (Eds.), *Handbook of International Research in Mathematical Education.* New York: Routledge.

Hogan, D., & Trudge, J. (1999). Implications of Vygotsky's Theory for Peer Learning. In A. O'Donnell and A. King (Eds.) *Cognitive Perspectives on Peer Learning*. New York: Routledge.

Jackiw, N. (2001). *The Geometer's Sketchpad.* Berkeley: Key Curriculum Press.

Jacobson, E. (2014). Using Covariation Reasoning to Support Mathematical Modeling. *Mathematics Teacher*, *107*(7), 515-519.

Kaiser, G. (2013). Introduction: ICTMA and the Teaching of Modeling and Applications. In R. Lesh, R. Galbraith, P. Haines, & A. Hurford (Eds.), *Modeling Students' Mathematical Competencies ICTM 13.* Netherlands: Springer.

Kaput, J. (1995). Long term algebra reform: Democratizing access to big ideas. In C. Lacampagne, W. Blair, & J. Kaput (Eds.), *The algebra initiative colloquium: Volume 1.* Washington, DC: U.S. Department of Education.

Kaput, J. (1995b). A research base supporting long term algebra reform? In *Proceedings of the Seventeenth Annual Meeting for the Psychology of Mathematics Education.* Retrieved from http://files.eric.ed.gov/fulltext/ED389539.pdf

Kaput, J. (1998). Transforming algebra from an engine of inequity to an engine of mathematical power by 'algebrafying' the K-12 Curriculum. In National Council of Teachers of Mathematics (Ed.), *The Nature and Role of Algebra in the K-14 Curriculum.* Washington DC: National Academy Press.

Kaput, J. (2007). What is algebra? What is algebraic reasoning? In J. Kaput, D. Carraher & M. Blanton (Eds.), *Algebra in the Early Grades.* New Jersey: Earlbaum.

Kaput, J., & Blanton, M. (2001). Algebrafying the elementary mathematics experience. In *The Future of the Teaching and Learning of Algebra. Proceedings of the 12th ICMI study conference.*

Keiran, C. (1996). The changing face of school algebra. In *8th International Congress on Mathematical Education – Selected Lectures.* Thales: S.A.E.M.

Keiran, C. (2007). Learning and Teaching Algebra at the Middle School Through College Levels: Building Meanings for Symbols and Their Manipulation. In F. Lester (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning: Volume 2.* New York: NCTM.

Ketterlin-Geller, L., and Chard, D. (2011). Algebra readiness for students with learning difficulties in grades 4-8: Support through the study of number. *Australian Journal of Learning Difficulties*, 16(1), 65-78.

Lannin, J., Ellis, A., & Elliot, R. (2011). *Adding it up: Helping children learn mathematics.* Washington DC: National Academy Press.

Lee, L., & Wheeler, D. (1987). *Algebraic thinking in high school students: Their conceptions of generalisation and justification.* Montreal: Concordia University Press.

Lehrer, R. and Lesh, R. (2013). Mathematical Learning. In I. Weiner, W. Reynolds & G. Miller (Eds). *Handbook of Psychology: Educational Psychology – Volume 7*. New Jersey: Wiley and Sons.

Lerman, S. (1996). Intersubjectivity in Mathematics Learning: A Challenge to the Radical Constructivist Paradigm?. *Journal for Research in Mathematics Education*, *27*(2), 133-50.

Lins, R., and Kaput, J. (2004). The Early Development of Algebraic Reasoning: The Current State of the Field. In K. Stacey, H. Chick, & M. Kendal (Eds.), *The Future of the Teaching and Learning of Algebra – The 12th ICMI Study.* Netherlands: Springer.

Lobarto, J. & Seibert, D. (2002). Quantitative reasoning in a reconceived view of transfer. *Journal of Mathematical Behaviour, 21(1)*, 87-116.

MacGregor, M., & Stacey, K. (1995). The effect of different approaches to algebra on students' perceptions of functional relationships. *Mathematics Education Research Journal, 7*, 69–85.

Mason, J. (2002). Generalization and algebra: Exploiting children's powers. In L. Hegarty (Ed.), *Aspects of teaching secondary mathematics: Perspectives on practice.* London: Routledge.

Mason, J. (2008). Making use of children's powers to produce algebraic thinking. In J. Kaput, D. Carraher & M. Blanton (Eds.), *Algebra in the Early Grades.* Erlbaum: New Jersey.

Mason, J. & Sutherland, R. (2002). *Key aspects of teaching algebra in schools.* London: QCA.

Moss, J., & London McNab, S. (2011). An approach to geometric and numeric patterning that fosters second grade students' reasoning and generalizing about functions and co-variations. In J. Cai & E. Knuth (Eds.), *Early algebraization: A global dialogue from multiple perspectives.* New York: Springer.

Mousoulides, N. & English, L. (2008). Modeling with data in Cypriot and Australian primary classrooms. In *Joint Meeting of the International Group and the North American Chapter of Psychology of Mathematics Education.* Mexico: PME.

Mulligan, J., Mitchelmore, M., English, L. D., & Crevensten, N. (2012). Evaluation of the "reconceptualising early mathematics learning" project. In *AARE 2012 Conference Proceedings. Australian Association for Research In Education.* Retrieved from http://eprints.qut.edu.au/57665/2/57665.pdf

National Council of Teachers of Mathematics (2008). *Algebra: What, when and for whom*. Retrieved from http://www.nctm.org/uploadedFiles/About_NCTM/Position_Statements/Algebra%20final%2092908.pdf.

Olive, J., Makar, K., Hoyos, V., Kor, L. K., Kosheleva, O., & Strasser, R. (2010). Mathematical knowledge and practices resulting from access to digital technologies. In C.Hoyles & J. Lagrange (2009), *Mathematics education and technology-rethinking the terrain: The 17th ICMI study (Vol. 13).* Springer Science & Business Media. Retrieved from http://eprints.ioe.ac.uk/16049/

Papert, S. (1980). *Mindstorms: Children, Computers, and Powerful Ideas.* New York: Basic Books.

Papert, S., & Harel, I. (1991). Situating constructionism. *Constructionism*, *36*, 1-11.

Pea, R. (1987). Logo Programming and Problem Solving: *Technical report: Number 12.* Retrieved from https://telearn.archives-ouvertes.fr/hal-00190546/document

Peirce, C. (1960). *Collected Papers of Charles Saunders Peirce: Volumes I and II.* Harvard: The Belnap Press of Harvard University.

Pelesko, J. (2014). Initial thoughts on the mathematical perspective. In J. Cai, M. Cirillo, J. Pelesko, R. Ferri, M. Borba, G. Stillman, L. English, G. Wake, G. Kaiser, & O. Kwon (2014). *Mathematical Modeling in School Education: Mathematical, Cognitive, Curricular, Instructional, and Teacher Education Perspectives: Proceedings of the 38th Conference of the International Group for the Psychology of Mathematics Education and the 36th Conference of the North American Chapter of the Psychology of Mathematics Education (Vol. 1). Vancouver: PM*

Polya, G. (1957). *How to solve it.* Princeton: Princeton University Press.

Punch, K. (2009). *Introduction to Research Methods in Education*. London: Sage.

Radford, L. (2003). Gestures, speech, and the sprouting of signs: A semiotic-cultural approach to students' types of generalization. *Mathematical Thinking and Learning, 5(1)*, 37–70.

Rakes, C., Valentine, I., McGatha, M., & Ronau, R. (2010). Methods of instructional improvement in algebra: a systematic review and meta-analysis. *Review of Educational Research, 80(3)*, 372-400.

Rivera, F. (2013). *Teaching and learning patterns in school mathematics: Psychological and pedagogical considerations*. Netherlands: Springer Science & Business Media.

Rivera, F. (2015). The distributed nature of pattern generalization. Retrieved from http://digibug.ugr.es/bitstream/10481/34989/1/Rivera2015PNA9%283%29Thedistributed.pdf

Rivera, F., & Rossi Becker, J. (2007). Abduction-induction (generalization) processes of elementary majors on figural patterns in algebra. *Journal of Mathematical Behaviour, 26,* 140-155.

Rivera, F. & Rossi Becker, J. (2008), Middle school children's cognitive perceptions of constructive and deconstructive generalizations involving linear figural patterns. *ZDM, 40,* 65-82.

Rogers, T., & McClelland, J. (2004). *Semantic cognition: A parallel distributed processing approach.* Retrieved from http://www.researchgate.net/profile/Timothy_Rogers/publication/10824868_The_parallel_distributed_processing_approach_to_semantic_cognition/links/0deec517feabc572dd000000.pdf

Russell, S. (1999). Mathematical reasoning in the elementary grades. *Developing mathematical reasoning in grades K-12*, *61(1)*.

Sacristan, A., Calder, N., Rojano, T., Santos-Trigo, M., Friedlander, A., Meissner, H., ... & Perrusquía, E. (2010). The Influence and Shaping of Digital Technologies on the Learning–and Learning Trajectories–of Mathematical Concepts. In *Mathematics Education and Technology-Rethinking the Terrain*. New York: Springer.

Sellares, J. & Toussaint, G. (2003). On the role of kinesthetic thinking in computational geometry. *International Journal of Education, Science and Technology,* 34(2), 219-237.

Simon, M. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal of Research in Mathematics Education, 26(2)*, 114-145.

Smith, E. (2008). Representational thinking as a framework for introducing functions in the elementary curriculum. In J. Kaput, D. Carraher & M. Blanton (Eds.), *Algebra in the early grades*. New York: Taylor & Francis Group.

Smith, E. & Thompson, P. (2007). Quantitative reasoning and the development of algebraic reasoning. In J. Kaput, D. Carraher, & M. Blanton (Eds.), *Algebra in the Early Grades*. NewYork: Erlbaum.

Stacey, K. (1989). Finding and using patterns in linear generalising problems. *Educational Studies in Mathematics*, 20,147–164.

Sutherland, R. (1991). Some unanswered research questions on the teaching and learning of algebra. *For the learning of mathematics, 11(3)* 40-46.

Teppo, A. (2015). Grounded Theory Methods. In A. Bikner-Ahsbahs, C. Knipping, & N. Presmeg (Eds.), *Approaches to Qualitative Research in Mathematics Education: Examples of Methodologies and Methods*. Germany: Springer.

Thompson, P. (2011). Quantitative Reasoning and Mathematical Modelling. In L. Hatfield, S. Chamberian & S. Belbase (Eds.), *New perspective and directions for collaborative research in mathematics education: WISDOMe Monographs - Volume 1.* Wyoming: University of Wyoming Press.

Van Reeuwijk, M., & Wijers, M. (1997). Students' Construction of Formulas in Context. *Mathematics Teaching in the Middle School, 2(4)*, 230-36.

Vygotsky, L. (1978). *Mind in Society: The Development of Higher Psychological Processes*, Harvard: Harvard University Press.

Vygotsky, L. (1981). The genesis of higher mental functions. In J. Wersch (Ed. and Trans.), *The Concept of Activity in Soviet Psychology*. New York: Sharpe.

Warren, E. (2005). Young children's ability to generalise the pattern rule for growing patterns. In *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics* Education: Volume 4. Retrieved from
http://emis.dsd.sztaki.hu/proceedings/PME29/PME29CompleteProc/PME29Vol4Mul_Wu.pdf#page=311

Watson, A., Jones K. & Pratt, D. (2013). *Key Ideas in Mathematics: Research-based guidance for ages 9-19*. Oxford: Oxford University Press.

Wilensky, U. (1991). *Abstract meditations on the concrete and concrete implications for mathematics education*. Massachusetts: Epistemology and Learning Group - MIT Media Laboratory.

Wilkie, K. (2014). Upper primary school teachers' mathematical knowledge for teaching functional thinking in algebra. *Journal of Mathematical Teachers' Education, 17,* 397-428.

Wilkie, K. J., & Clarke, D. (2015). Pathways to Professional Growth: Investigating Upper Primary School Teachers' Perspectives on Learning to Teach Algebra. *Australian Journal of Teacher Education*, *40(4),* 87-118.

Yackel, E., & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for research in mathematics education, 27(4),* 458-477.